Discussion

Considering that Figs. 2 and 3 are based primarily on test data that give launch speed rather than v_0‡ and that offer no clue as to variations of the angle θ in the Froude-number-related parameter $gD_0 \sin\theta/v_0^2$, the data of the figures are sufficiently well correlated to indicate the general validity of the model law set forth in Eq. (7).

Since the test data used (cf. Table 1) are for cases well below Mach 1, compressibility effects do not enter into any of the scatter observed in Figs. 2 and 3. However, in the derivation of Eq. (7) it was assumed that C_D could be treated solely as a characteristic of the parachute, which may not be the case. That is, although the relatively high Reynolds numbers of Table 1 appear to justify neglect of viscous effects, the Reynolds numbers of the table are based on parachute diameter D_0, which does not appear to be the best choice of a characteristic length for flow of air through the canopy. A better choice would be a Reynolds number based on some characteristic dimension of the canopy material, and the resulting (lower) Reynolds numbers might well be significant with respect to opening shock.

Equation (7) indicates that both $\rho D_0^3/M$ and $gD_0 \sin\theta/v_0^2$ must be held constant for dynamical similarity in free-flight testing of parachute opening shock. It is, in fact, physically possible to maintain simultaneously both of these parameters at their correct values. The model law and data presented herein also indicate that proper adjustment of the test variables may be used to test opening shock at low altitude on a parachute system intended for use at high altitude, to extrapolate opening shock test data on scale models to full scale, and to test in the earth's atmosphere parachute systems intended for use in other planetary atmospheres. However, it is obvious that more and better test data are required to verify completely any model law for incompressible flow parachute opening shock, with data particularly required at $\rho D_0^3/M$ numbers from 0 to, say, 5.

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‡ It is difficult to obtain a value for v_0 corresponding to a definite radius r_0 for the chute in an early stage of inflation, whereas it is a trivial matter to report the airspeed at which a packed parachute is launched from an aircraft or test vehicle. The result is, unfortunately, that only the launch speed is usually reported.

Capillary Stability in an Inverted Rectangular Channel for Free Surfaces with Curvature of Changing Sign

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Nomenclature

B = Bond number
F = equilibrium interface
x, y = nondimensional Cartesian coordinates
Δ = $\pm$ curvature
ϵ = variational parameter
η = perturbation to F
θ = contact angle
λ = undetermined parameter
ψ = angle between tangent to F and horizontal
$(\ )'$ = differentiation with respect to x

Introduction

In a previous paper,[1] an investigation of the stability of an incompressible inviscid fluid contained in an inverted rectangular channel was performed mathematically with the effects of surface tension taken into account. The purpose was to determine the effect of contact angle and Bond number on the stability of such a fluid configuration in a low-gravity environment. The investigation was restricted to those fluid-gas equilibrium interfaces for which the curvature does not change sign. The analysis is extended in this report to include equilibrium interfaces for which the curvature does change sign. The omission of such interfaces in the original analysis is shown to be justified because they are all dynamically unstable. In addition, the details omitted in Ref. 1 of the calculation of the critical Bond number for equilibrium interfaces for which the curvature does not change sign are also presented.

Formulation

The differential equation for an equilibrium free surface $y = F(x)$ (see Fig. 1) is given by Eq. (3) of Ref. 1 as

$$\{F'/[(1 + F'^2)]^{1/2}\}' + BF - \lambda = 0 \qquad -1 < x < 1 \quad (1)$$

The boundary conditions are given by Eq. (4) of Ref. 1 and are

$$F'(1) = -F'(-1) = \cot\theta \qquad (2)$$

where θ is the contact angle between the fluid and the wall. Only values of θ for which $0 < \theta < \pi/2$ (wetting fluid) will be considered, since, as was shown in Ref. 1, the problem for $\pi/2 < \theta < \pi$ (nonwetting fluid) can be formulated in terms of an equivalent problem for $0 < \theta < \pi/2$.

Equation (1) can be solved in terms of the parameter

$$\psi = \tan^{-1}F' \qquad (2a)$$

the slope of the surface, to yield

$$B(F^2/2) - \lambda F = \cos\psi - C$$

where F is now a function of ψ. The boundary conditions, Eq. (2), become $\psi = \pm(\pi/2 - \theta)$ at $x = \pm 1$, respectively. The constant of integration C may be conveniently evaluated by choosing the origin of coordinates properly. The coordinates shown in Fig. 1 of Ref. 1 were chosen so that the origin would lie on the fluid-vapor interface, halfway between the walls. It is more convenient here to choose the x axis to

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intersect the interface when $\psi = 0$ rather than when $x = 0$. That is, $F = 0$ when $\psi = 0$ and $x = x_0$(x_0 is not necessarily 0, see Fig. 1). The boundary conditions require that $\psi = 0$ at least once by continuity, but ψ may equal zero more than once, in which case any particular point where $\psi = 0$ and the curvature is positive (concave downward) may be singled out to define where $F = 0$. With this choice of coordinates, the constant of integration C is one, so that

$$(BF^2/2) - \lambda F = \cos\psi - 1$$

or

$$F = (\lambda \pm \Delta/B) \tag{3}$$

where

$$\Delta = [\lambda^2 + 2B(\cos\psi - 1)]^{1/2} \tag{4}$$

Substitution into Eq. (1) shows that $\mp \Delta$ is the curvature. Notice that the value of $C = 1$ is not only compatible with $F = 0$ at $\psi = 0$, but also with $F = 2\lambda/B$ at $\psi = 0$, depending on whether one takes the lower or upper sign for the square root.

The corresponding solution for x passing through $x = x_0$ when $\psi = 0$ is obtained from Eqs. (2a) and (3). It is

$$x = x_0 \mp \int_0^{\psi} \frac{\cos\psi\, d\psi}{\Delta} \tag{5}$$

The value of x_0 can be determined from the boundary conditions once a choice is made concerning the sign.

In Ref. 1 (as in the first reference of Ref. 1), only the solution taking the lower sign is considered, since it is the one passing through the required value $F = \psi = 0$. The solution taking the upper sign is not considered on the grounds that it does not pass through $F = \psi = 0$. The ensuing analysis in Ref. 1 then determines that $x_0 = 0$, and, on this basis, the critical Bond number is found. Notice, however, that for values of ψ equal to $\pm\psi_0$ where the curvature vanishes,

$$\psi_0 = \cos^{-1}[1 - (\lambda^2/2B)] \qquad 0 \leqq \psi_0 < \pi \tag{6}$$

it is possible to join continuously the two branches of the solution, one having the upper sign and the other the lower sign in Eq. (3). Since the curvature vanishes at these points, the derivatives up to the second would also join continuously. Equation (6) shows that joining can take place (for real values of ψ) only if $B \geqq \lambda^2/4$. In this paper, values of $B \geqq \lambda^2/4$ are considered, and equilibrium solutions made up of joined branches are allowed. The purpose is to examine the stability of such joined solutions.

Stability Analysis for Joined Interfaces

The stability of the equilibrium interfaces consisting of joined solutions can be investigated by considering the second variation in the same manner described in Ref. 1 for unjoined solutions. The second variation for trial functions of the form $y = F(x) + \epsilon\eta(x)$, where $F(x)$ is the equilibrium solution, ϵ is a parameter, and $\eta(x)$ is the perturbing function, is given by Eqs. (5) and (6) of Ref. 1. Equation (7) of Ref. 1 is the Jacobi equation, which must be satisfied by $\eta(x)$ to make the second variation vanish. It can be written in the form

$$[\eta'/(1 + F'^2)^{3/2}]' + B\eta = 0 \tag{7}$$

In the present case, care must be taken to use the appropriate signs in each of the branches of F.

No matter which sign is chosen for F and x, however, the relationship

$$dF/dx = \tan\psi \tag{8}$$

still holds, and Eq. (7) becomes

$$(d/dx)[\cos^3\psi(d\eta/dx)] + B\eta = 0$$

upon substitution of Eq. (8).

This equation, when rewritten in terms of derivatives with respect to ψ, yields

$$\frac{(d/d\psi)[\cos^3\psi(d\eta/d\psi)]}{(dx/d\psi)^2} - \frac{(d^2x/d\psi^2)\cdot[\cos^3\psi(d\eta/d\psi)]}{(dx/d\psi)^3} + B\eta = 0 \tag{9}$$

Substitution of Eq. (5) into Eq. (9) shows that Eq. (9) is independent of whether the upper or lower sign is used in the solution for x. Thus, Jacobi's equation is unchanged when joined solutions, as well as nonjoined solutions, are considered. Hence, the least eigenvalue, which is the critical Bond number, is also unchanged. Therefore, the stability criterion for the critical Bond number is unaltered from that given in Ref. 1.

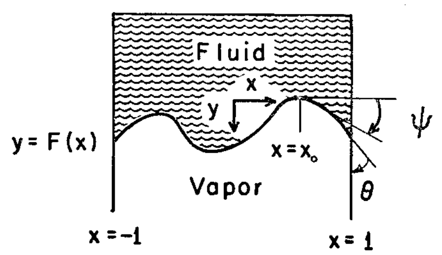

Fig. 1 Geometric configuration.

It is now shown that, for Bond numbers less than the critical one, only the simple unjoined interfaces discussed in Ref. 1 can be stable; the joined interfaces are always unstable. To do this, it is sufficient to show that for joined interfaces a conjugate point to one of the end points $x = \pm 1$ lies within the interval $-1 < x < 1$; or, in other words, if the general solution of Eq. (7) is made to satisfy nontrivially one of the boundary conditions Eq. (2) in Ref. 1 [that $d\eta/dx = 0$ at $x = -1$ or $x = +1$] then $d\eta/dx$ will also vanish for some x in this interval. Such a property makes it possible to choose a function that makes the second variation negative.

For Δ to be real, ψ must lie within the limits $-\psi_0 \leqq \psi \leqq \psi_0$. Thus, if F is to be an equilibrium interface satisfying the boundary conditions, Eq. (2), then $\pi/2 - \theta \leqq \psi_0$. If, in addition, there is an interior joining point, then $\psi = \pm\psi_0$ for some point in $-1 < x < 1$. Then continuity in ψ requires that ψ must take on at least one of the values $-(\pi/2 - \theta)$ or $(\pi/2 - \theta)$ in the interior. That is, in order for there to be a joining point, ψ must assume one of its boundary values at least once in the interior. However, this means that a solution of Eq. (7), for which $d\eta/dx$ vanishes at a boundary point, also has $d\eta/dx$ vanish at an interior point because, by use of $d\psi/dx$ as calculated from Eq. (5), it follows that

$$(d\eta/dx) = \mp (\Delta/\cos\psi)(d\eta/d\psi)$$

a function of ψ alone. Thus, joined interfaces cannot provide an extremal solution, and hence they cannot be stable.

Conclusion

The foregoing analysis shows that equilibrium interfaces possessing a curvature that changes sign are always unstable and that the only possible stable interfaces are those with a curvature that does not change sign. Thus, the stability criteria for the critical Bond number derived in Ref. 1 by consideration of only those interfaces possessing curvature of constant sign are valid in general.

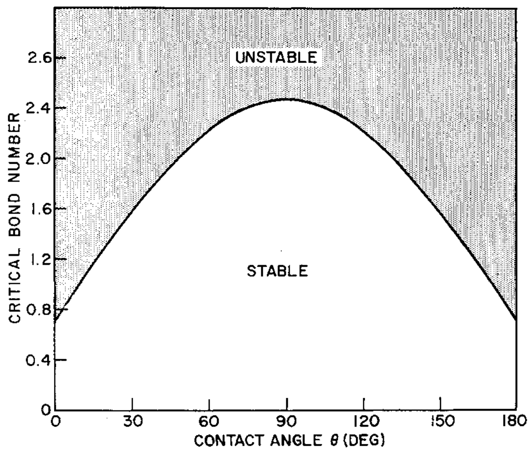

Fig. 2 Critical Bond number as a function of contact angle.

Appendix: Calculation of the Critical Bond Number for the Stability of Equilibrium Interfaces

The critical Bond number can be calculated by first finding the general solution of Eq. (9) for η as a function of ψ. Substitution of $d\psi/dx$, as calculated from Eq. (5), into Eq. (9) yields

$$\frac{d}{d\psi}\left(\cos^3\psi\,\frac{d\eta}{d\psi}\right) + \sin\psi\,\cos^2\psi\left(1 - B\,\frac{\cos\psi}{\Delta^2}\right)\frac{d\eta}{d\psi} + B\eta\,\frac{\cos^2\psi}{\Delta^2} = 0$$

The general solution is

$$\eta = C\tan\psi + D[(\lambda/B\Delta) + \tan\psi\cdot I(\psi)]$$

where

$$I(\psi) = -\int_0^{\psi}\frac{\lambda^2\cos\psi\,d\psi}{\Delta^3}$$

It is always required that $-\psi_0 \leqq \psi \leqq \psi_0$ in order that Δ be real. The derivative of η with respect to ψ is given by $d\eta/d\psi = \sec^2\psi[C + DI(\psi)]$, so that the derivative with respect to x is

$$\frac{d\eta}{dx} = \frac{d\eta}{d\psi}\,\frac{d\psi}{dx} = \mp\,\Delta\sec^3\psi\cdot[C + DI(\psi)]$$

Application of the boundary conditions at $x = \pm 1$ yields

$$\Delta_1\sec^3\psi_1[C - DI(\psi_1)] = 0$$

$$\Delta_1\sec^3\psi_1[C + DI(\psi_1)] = 0$$

where $\psi_1 = \pi/2 - \theta$, and Δ_1 is the value of Δ when ψ equals ψ_1. Since for $0 < \psi_1 < \pi/2$ neither $\sec^3\psi_1$ nor $I(\psi_1)$ can vanish, the equations can be satisfied only if $\Delta_1 = 0$, and the product $\Delta_1 I(\psi_1)$ may, in general, be nonzero so that D must also be chosen zero. The requirement that $\Delta_1 = 0$ implies that

$$\lambda^2 = 2B(1 - \cos\psi_1) = 2B(1 - \sin\theta)$$

When this relationship is used in Eq. (5) and the boundary conditions that $\psi = \pm(\pi/2 - \theta)$ at $x = \pm 1$, respectively, are applied, it follows that $x_0 = 0$, and that the critical Bond number relationship is

$$\frac{1}{2}\left\{\int_0^{(\pi/2-\theta)}\frac{\cos\psi\,d\psi}{(\cos\psi - \sin\theta)^{1/2}}\right\}^2 = B$$

which is Eq. (19) of Ref. 1. Figure 2 of Ref. 1 is reproduced here which is a graph of the foregoing relationship giving the critical Bond number as a function of contact angle.

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